The number of monounary algebras

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Abstract. In this note we give an asymptotic estimate for the number of monounary algebras of given size.

1. Introduction

A monounary algebra is an algebra with a single unary operation. The theory of monounary algebras is well-developed, for a recent monograph see [2]. Let $\mathcal{A} = (A, f)$ be a monounary algebra. The function f defines a directed graph on A. Let $G_A = (A, E)$, the vertex set is A and the edges are $E = \{(a, f(a)) \mid a \in A\}$. In G_A every vertex has outdegree 1, and every graph G with outdegree 1 defines a monounary algebra on its vertex set, where $f(a)$ is the single vertex such that $(a, f(a))$ is an edge in G. Hence, there is a bijection between monounary algebras and directed graphs, where each vertex has outdegree 1. At first, we investigate the number of finite connected non-isomorphic monounary algebras.

Theorem 1.1. *Let C_n denote the number of connected non-isomorphic monounary algebras of size n. Then there is an $\alpha > 1$ such that $\log_\alpha C_n \sim n$.*

Proof. A directed graph corresponding to a connected monounary algebra is a directed cycle with a (possibly one-element) rooted tree at each vertex of the cycle. If there is a loop in the graph of the algebra, we say that the length of the cycle is 1. Each edge of each rooted tree is directed towards the cycle, and the cycle consists of the roots of the trees. Hence, a connected graph of a monounary algebra of size n is built up from a cycle of length k, where $1 \le k \le n$ and to each vertex of the cycle we glue a rooted tree such that the sum of the sizes of the rooted trees is n. Every connected monounary algebra of size n can be obtained in this way. Note that these graphs are not necessarily non-isomorphic. Naturally, to each rooted tree there corresponds a unique monounary algebra with a one-element cycle. Let T_n denote the number of rooted trees on n vertices. By the above arguments we have

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$$T_n \leq C_n \leq \sum_{k=1}^{n} \sum_{i_1+\cdots+i_k=n} T_{i_1} T_{i_2} \cdots T_{i_k}. \tag{1}$$

Let $T_0 = C_0 = 1$. Let $\psi(x) = \sum\limits_{n=1}^{\infty} T_n x^{n-1}$ and $\Gamma(x) = \sum\limits_{n=0}^{\infty} C_n x^n$. Now Γ is the generating function of the number of connected monounary algebras, and $x\psi(x)+1$ is the generating function of the number of rooted trees. By formula (1) the following holds:

$$x\psi(x)+1 \leq \Gamma(x) \leq 1+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} \sum_{i_1+\cdots i_k=n} T_{i_1} T_{i_2} \ldots T_{i_k}\right) x^n = \sum_{n=0}^{\infty} (x\psi(x))^n \tag{2}$$

In [3] the function $\psi(x)$ is well analyzed. It is proved that there is a constant c_T such that $T_n \sim c_T \alpha^n n^{-\frac{3}{2}}$, where $\alpha \sim 2.955765$. Moreover, the power series $\psi(x)$ has radius of convergence $\frac{1}{\alpha}$ for the same α, and $\psi(\frac{1}{\alpha}) = \alpha$, hence $\frac{1}{\alpha}\psi(\frac{1}{\alpha}) = 1$ holds. As $\sum\limits_{n=0}^{\infty} (x\psi(x))^n = \dfrac{1}{1-x\psi(x)}$ is strictly monotonically increasing in $\mathbb{R}^+$, the radius of convergence of this power series is the unique positive solution of the equation $x\psi(x) = 1$, which is $\frac{1}{\alpha}$. Therefore for each power series in (2) the radius of convergence is $\frac{1}{\alpha}$. This holds for $\Gamma(x)$, as well. Thus for the coefficients of $\Gamma(x)$ we have $\limsup\limits_{n\to\infty} \sqrt[n]{C_n} = \alpha$. From $T_n \leq C_n$ we obtain $\liminf\limits_{n\to\infty} \sqrt[n]{C_n} \geq \alpha$, thus $\lim\limits_{n\to\infty} \sqrt[n]{C_n} = \alpha$, and $\log_\alpha C_n \sim n$ is gained.

□

Note that asymptotic tree counting techniques in which generating functions were applied, were investigated in more detail also in [1].

Theorem 1.2. *Let M_n denote the number of monounary algebras of size n. Then $\log_\alpha M_n \sim n$.*

Proof. A monounary algebra is the disjoint union of connected monounary algebras. Let $\mathcal{A}$ be a monounary algebra of size n consisting of p_i connected algebras of size i. Then $\sum\limits_{i=1}^{n} ip_i = n$. Note that p_i can be 0. The number of ways of picking k indistinguishable objects of type p is the coefficient of x^k in the generating function $(1-x)^{-p}$. Hence, the number of ways picking p_i indistinguishable connected monounary algebras of type C_i is the coefficient of x^{ip_i} in $(1-x^i)^{-C_i}$. Thus the generating function for the number of monounary algebras is

$$1+\sum_{n=1}^{\infty} M_n x^n = \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{C_k}} \tag{3}$$

For $x > 0$ this series converges if and only if

$$\log\left(\prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{C_k}}\right) = \sum_{k=1}^{\infty} -C_k \log(1-x^k)$$

is convergent. If $x \in (0, \frac{1}{\alpha})$ then $x^k \in (0, \frac{1}{\alpha})$ for all $k \geq 1$. As $\log x$ is concave we have $\log(1-t) \geq -ct$ for all $t \in (0, \frac{1}{\alpha})$ with $c = -\alpha \log(1 - \frac{1}{\alpha})$. Therefore $\sum\limits_{k=1}^{\infty} -C_k \log(1-x^k) \leq \sum\limits_{k=1}^{\infty} C_k c x^k = c(\Gamma(x) - 1)$. Hence this power series – and consequently $\sum\limits_{n=0}^{\infty} M_n x^n$ – is convergent in $(0, \frac{1}{\alpha})$. This yields $\limsup\limits_{n\to\infty} \sqrt[n]{M_n} \leq \alpha$. The lower bound $\liminf\limits_{n\to\infty} \sqrt[n]{M_n} \geq \alpha$ can be derived from $C_n \leq M_n$. Thus $\lim\limits_{n\to\infty} \sqrt[n]{M_n} = \alpha$ and $\log_\alpha M_n \sim n$.

□

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